New Formulae for 3DoF Space Charge Field

Y. Alexahin (FNAL APC)

There is a large number of programs for space charge effect simulations using various methods to calculate the space charge field which can be divided in two main groups: the first uses actual particle distribution obtained on the preceding step of simulations (e.g. PIC, multipole expansion) while the other relies on a smooth approximation (e.g. Gaussian) of particle distribution and analytical formulas for the field.

Here we present new formulas long bunches with charge density described by

$$\rho(x,y,z,t) = \frac{\lambda(z-v_0 t)}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right). \tag{1}$$

where λ is linear charge density (not necessarily Gaussian).

For a long bunch, $\sigma_z >> \max(\sigma_x, \sigma_y)$, we can neglect variation of $\lambda(z)$ and use formulas for two-dimensional charge distribution. Then for the transverse electric field the well-known Basetti-Erskine formula [1] can be used.

In this approximation the transverse field (and the associated kick) is proportional to the charge density at the particle location, $\lambda(z)$. For symplecticity of the 6D transfer map it must be complemented by a longitudinal kick dependent on the transverse coordinates.

The symplecticity will be guaranteed if the field components are derived from the same potential. In the present report we give the space charge potential of a (transversely) Gaussian bunch in a convenient form for numerical calculations and also provide an alternative to the Erskine-Basetti formula. This approach was successfully used in the beam-beam effect analyses [2].

In the long bunch approximation the time retardation can also be neglect to give direct space charge potential in the form[1]

$$\varphi(x,y,z,t) \cong \frac{\lambda(z-v_0 t)}{2\pi\sigma_x\sigma_y}\iint G(x-x', y-y')\exp\left(-\frac{x'^2}{2\sigma_x^2}-\frac{y'^2}{2\sigma_y^2}\right)dx'dy'\,. \tag{2}$$

where the Green function can be presented as

$$G = -\ln(\Delta x^2+\Delta y^2) = \frac{r}{\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\exp\left(ik_1\frac{\Delta x}{\sigma_x}+ik_2\frac{\Delta y}{\sigma_y}\right)\frac{dk_1 dk_2}{r^2k_1^2+k_2^2}\quad , \tag{3}$$

$r = \sigma_y/\sigma_x$, so far this choice being arbitrary.

Performing in eq.(2) integration by transverse variables we obtain

$$\varphi(x,y,z,t) \cong \lambda(z-v_0 t)\frac{r}{\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\exp\left(ik_1\frac{x}{\sigma_x}+ik_2\frac{y}{\sigma_y}-\frac{k_1^2+k_2^2}{2}\right)\frac{dk_1 dk_2}{r^2k_1^2+k_2^2}. \tag{4}$$

Making use of the formula

---

[1] Gaussian units are used. To convert to SI units the r.h.s. should be divided by $4\pi\varepsilon_0$

$$\frac{1}{r^2k_1^2+k_2^2}=\int_0^\infty e^{-(r^2k_1^2+k_2^2)\tau}d\tau \tag{5}$$

with subsequent integration by $k_{1,2}$ and finally setting $\tau = (t^{-1}-1)/2r^2$ we get

$$\varphi(x,y,z,t)\cong\lambda(z-v_0t)\int_0^1 \exp\left(-\frac{x^2t}{2\sigma_x^2}-\frac{y^2r^2t}{2\sigma_y^2[1+(r^2-1)t]}\right)\frac{dt}{t\sqrt{1+(r^2-1)t}}\,. \tag{6}$$

We deliberately have chosen an asymmetric form to make eq.(6) easier to treat at least in some cases (see later).

Thus far we have not paid attention to the logarithmic divergence of the integral (6) at t=0: it disappears in the formula for the transverse field. But if we need the longitudinal field we have to eliminate this divergence.

The simplest way to regularize the potential (6) is by subtracting unity from the exponential in the integrand to obtain

$$\varphi(x,y,z,t)\cong\lambda(z-v_0t)\int_0^1\left\{\exp\left(-\frac{x^2t}{2\sigma_x^2}-\frac{y^2r^2t}{2\sigma_y^2[1+(r^2-1)t]}\right)-1\right\}\frac{dt}{t\sqrt{1+(r^2-1)t}}\,. \tag{7}$$

This regularization does not affect the transverse field at all and makes the potential (hence the longitudinal field) identically zero on the beam axis (x=y=0). If necessary longitudinal and transverse wakes can be added on top of the field obtained from potential (7).

Power series

Integral (7) as well as its derivatives w.r.t. the transverse coordinates can be computed numerically but for small offsets

$$\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\le n^2\,, \tag{8}$$

$n \le$ 4-5, a power expansion can be used. Its coefficients are integrals which can be expressed via the Gauss hypergeometric series:

$$B(m,l;a)\equiv\int_0^1\frac{t^m dt}{(1+at)^{l+1/2}}=\frac{1}{m+1}\ {}_2F_1(m+1,l+1/2;m+2;-a)\,. \tag{9}$$

with $a = r^2 - 1$. Then we have for the potential

$$\varphi=\lambda\sum_{m=0}^{M}\sum_{l=0}^{L}\left(-\frac{1}{2}\right)^{m+l}\left(\frac{x}{\sigma_x}\right)^{2m}\left(\frac{ry}{\sigma_y}\right)^{2l}\frac{1-\delta_{m+l,0}}{m!l!}B(m+l-1,l;a)\,. \tag{10}$$

where $\delta_{m,n}$ is the Kronecker delta: $\delta_{m,n}$ =1 if m=n and $\delta_{m,n}$ =0 otherwise.

The following recurrence relations can be used to compute coefficients (9)

$$\begin{aligned} B(m,l;a) &= \frac{1}{m+1}\left[\frac{1}{(1+a)^{l-1/2}} - a(m-l+3/2)B(m+1,l;a)\right], \\ B(m,l+1;a) &= \frac{1}{l+1/2}\left[\frac{1}{(1+a)^{l+1/2}} - (m-l+1/2)B(m,l;a)\right]. \end{aligned} \tag{11}$$

For $|a| < 1$ the first relation should be used as it is presented, in the descending order in m, while for $a = r^2 - 1 > 1$ it must be used in the descending order in m. So to obtain the whole set of coefficients just one integral has to be computed (or a hypergeometric function if $|a| < 1$).

These coefficients determine not only the potential but also the electric field components, so that all necessary quantities can be obtained simultaneously.

With M=50 and L=M-m in eq. (10) a better than 6-digits precision is obtained for n = 5.6 in the case $a < 0$. For $a > 1$ the error can be by an order of magnitude larger, so it is better to use formulas with interchanged x and y if $\sigma_y > \sigma_x$.

Asymptotic expansion

For distant halo particles ($n > 5$ in eq. (8)) we can employ expansion in inverse powers of coordinates but also use power series to calculate the integral in (7) in the close vicinity of t=0.

Let us define parameter

$$u = n^2 \Bigg/ \left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right), \tag{11}$$

with chosen n and, if $u < 1$, divide integration interval in (7) in two parts:

$$\int_0^1 dt = \int_0^u dt + \int_u^1 dt\,, \tag{12}$$

using power series in the first integral:

$$\int_0^u dt = \sum_{m=0}^{M}\sum_{l=0}^{L}\left(-\frac{u}{2}\right)^{m+l}\left(\frac{x}{\sigma_x}\right)^{2m}\left(\frac{y}{\sigma_y}\right)^{2l}\frac{1-\delta_{m+l,0}}{m!l!}\,B(m+l-1,l;\,au)\,. \tag{13}$$

The part of the second integral which contains exponential can be found either by direct numerical integration or as an asymptotic expansion by integrating by parts. Here we assume that $|x|/\sigma_x \geq |y|/\sigma_y$, otherwise we will have to start with eq. (6) with interchanged x and y and understand r as σ_x/σ_y.

The integration by parts in (6) gives

$$\int_u^1 \exp\left(-\frac{x^2 t}{2\sigma_x^2}-\frac{y^2 t}{2\sigma_y^2}\right)Y(t)dt = -\exp\left(-\frac{x^2 t}{2\sigma_x^2}-\frac{y^2 t}{2\sigma_y^2}\right)\sum_{m=1}^{\infty}\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)^{-m} Y^{(m-1)}(t)\Big|_u^1 ,$$

$$Y(t)=\exp\left[-\frac{y^2 at(1-t)}{2\sigma_y^2(1+at)}\right]\Big/\left(t\sqrt{1+at}\right), \quad a=r^2-1. \tag{14}$$

Analytical expressions for the derivatives of function $Y(t)$ can be easily found with the help of programs for symbolic computations such as *Mathematica*. To obtain with n = 4 the same 6-digits precision of the total integral typically about three terms should be retained in the sum. With larger n in eq. (11) this addition may be not necessary at all.

The remaining part of the second integral evaluates analytically:

$$-\int_u^1 \frac{dt}{t\sqrt{1+at}} = \log\frac{(1+\sqrt{1+at})^2}{t}\Bigg|_u^1 . \tag{15}$$

References

[1] M. Basetti, G.A. Erskine, CERN-ISR-TH/80-06 (1980)

[2] Y. Alexahin, FERMILAB-TM-2148 (2001).